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Excitation of ordinary and extraordinary focus wave modes in a uniaxial crystal

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Abstract. We proved in previous works that two kinds of focus wave modes, ordinary and extraordinary, can propagate in a dielectric uniaxial crystal. We now discuss the conditions for reflection and refraction of focus wave modes incident from vacuum on such a crystal. Then we look for the amplitudes of the reflected, refracted, ordinary and extraordinary focus wave modes.

1. Introduction

In an attempt to transmit energy in a non-conventional way, people paid attention, some years ago, to particular solutions of Maxwell's and the wave equations, i.e. the so-called focus wave modes (FWMs) [1, 2] which have the property of propagating without dispersion. In fact, FWMs are a special class of solutions called relatively undistorted progressing waves by Courant and Hilbert [3]: these waves keep their identity established by their phase throughout their lifetime with an amplitude which decreases with time. Many works [4–8] have been devoted to the physical properties of FWMs which are modulated Gaussian beams and appear as the relativistic generalization of conventional Gaussian beams [9].

On the other hand, the importance especially in optics of anisotropic media such as dielectric crystals, plasma, ferrite and so on, is well known [10]. So, it is natural to inquire how FWMs behave in these media and recently [11] we have analysed the kinds of FWMs which are able to propagate in uniaxial anisotropic crystals. We continue this work here by investigating what happens to a FWM incident from free space on a crystal and first we define more accurately the problem to be tackled.

With coordinates along the principal axes of the permittivity tensor, the uniaxial anisotropic dielectrics are defined by the constitutive relations

$$D_{x,y} = \varepsilon E_{x,y} \qquad D_z = \eta E_z \qquad \boldsymbol{B} = \mu \boldsymbol{H} \tag{1}$$

in which for monochromatic fields, ε, η, μ, are constant scalars. A monochromatic FWM incident from free space on the face $z = 0$ of a slab made of a material with the constitutive relations (1), normal to the optical axis oz, will give rise to a transmitted and a reflected field. We first consider the nature of the transmitted field, confining our attention to finding the direction of propagation of the disturbance within the crystal and outside the opposite face $z = d$ of the slab. Then, expressions for the amplitude ratios (corresponding to the Fresnel formulae) of the waves excited in the crystal are investigated.

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To find the conditions for reflection and refraction, one has just to work with the phase function $\exp(\mathrm{i}\Omega)$, $\mathrm{i} = \sqrt{1}$, in which Ω, as proved in [3], is a solution of the characteristic equation of Maxwell's equations. In free space, this characteristic equation is [3]

$$(\partial_x\Omega)^2 + (\partial_y\Omega)^2 + (\partial_z\Omega)^2 - c^{-2}(\partial_t\Omega)^2 = 0 \tag{2}$$

in which c is the velocity of light, while in the crystal (1) we have [11] for the extraordinary wave with $n^2 = \varepsilon\mu$, $m^2 = \eta\mu$

$$m^{-2}[(\partial_x\Omega)^2 + (\partial_y\Omega)^2] + n^{-2}(\partial_z\Omega)^2 - c^{-2}(\partial_t\Omega)^2 = 0 \tag{3a}$$

and for the ordinary wave

$$n^{-2}[(\partial_x\Omega)^2 + (\partial_y\Omega)^2 + (\partial_z\Omega)^2] - c^{-2}(\partial_t\Omega)^2 = 0. \tag{3b}$$

So, we have first to find the FWM solutions of equations (2), (3*a*) and (3*b*) and then to match these solutions on the interface $z = 0$ between free space and the crystal to satisfy the continuity of the transverse components of the wavevector $\boldsymbol{k} = \mathrm{grad}\,\Omega$, that is

$$(\partial_x\Omega)_{z=0-} = (\partial_x\Omega)_{z=0+} \qquad (\partial_y\Omega)_{z=0-} = (\partial_y\Omega)_{z=0+}. \tag{4}$$

These boundary conditions give the usual Descartes–Snell law when the phase Ω is a linear function of t and $\boldsymbol{x}$ (harmonic plane waves) which is not the case for FWMs.

For the sake of simplicity, we start this investigation with the case of a phase not depending on one coordinate, say y. Then, one has a two-dimensional problem corresponding to the propagation of transverse magnetic (TM) and transverse electric (TE) electromagnetic FWMs and it was proved in [11] that the phase of TM and TE waves is a solution of equations (3*a*) and (3*b*), respectively (with of course $\partial_y\Omega = 0$).

2. Refraction of TM and TE focus wave modes

We use the subscripts i and t to denote quantities connected with the incident and refracted fields, respectively. The phase of TM and TE FWMs propagating in free space along a direction making an angle u_i with the z-axis and satisfying equation (2) is [1, 2, 12]

$$\Omega_\mathrm{i} = k_\mathrm{i}[ct - Z_\mathrm{i} - X_\mathrm{i}^2(a + ct + Z_\mathrm{i})^{-1}] \tag{5}$$

$$Z_\mathrm{i} = z\cos u_\mathrm{i} + x\sin u_\mathrm{i} \qquad X_\mathrm{i} = x\cos u_\mathrm{i} - z\sin u_\mathrm{i} \tag{5a}$$

in which k_i is the wavenumber, while $a = \mathrm{i}w$ is a purely imaginary parameter with w playing the role of the beam half-width at the origin of a Gaussian beam.

For a TM FWM propagating within the crystal in a direction making the angle u_t with respect to the principal axis of the crystal (1), the phase Ω_t solution of equation (3*a*) is [11]

$$\Omega_\mathrm{t} = k_\mathrm{t}[ct - Z_\mathrm{t} - X_\mathrm{t}^2(a + ct + Z_\mathrm{t})^{-1}] \tag{6}$$

$$Z_\mathrm{t} = nz\cos u_\mathrm{t} + mx'\sin u_\mathrm{t} \qquad X_\mathrm{t} = mx'\cos u_\mathrm{t} - nz\sin u_\mathrm{t} \tag{6a}$$

in which, for reasons to become clear soon, we use a new system of coordinates (x', z). Changing m into n in (6*a*) gives the phase of the TE FWM solution of equation (3*b*).

To discuss the first condition (4) (the second one is satisfied trivially), we use (5*a*) to write (5)

$$\Omega_\mathrm{i} = k_\mathrm{i}[a(ct - z\cos u_\mathrm{i} - x\sin u_\mathrm{i}) + c^2t^2 - x^2 - z^2](a + ct + z\cos u_\mathrm{i} + x\sin u_\mathrm{i})^{-1} \tag{7}$$

and a simple calculation gives

$$(\partial_x \Omega_{\rm i})_{z=0} = -k_{\rm i}(2x + a \sin u_{\rm i})(a + ct + x \sin u_{\rm i})^{-1} \\ -k_{\rm i} \sin u_{\rm i}[a(ct - x \sin u_{\rm i}) + c^2t^2 - x^2](a + ct + x \sin u_{\rm i})^{-2}. \qquad (7a)$$

Similarly, from (6) and (6*a*)

$$\Omega_{\rm t} = k_{\rm t}[a(ct - nz \cos u_{\rm t} - mx' \sin u_{\rm t}) \\ + c^2t^2 - m^2x'^2 - n^2z^2](a + ct + nz \cos u_{\rm t} + mx' \sin u_{\rm t})^{-1} \qquad (8)$$

and

$$(\partial_x \Omega_{\rm t})_{z=0} = -k_{\rm t}m(2mx' + a \sin u_{\rm t})(a + ct + mx' \sin u_{\rm t})^{-1} \\ -km \sin u_{\rm t}[a(ct - mx' \sin u_{\rm t}) + c^2t^2 - m^2x'^2](a + ct + mx' \sin u_{\rm t})^{-2}. \qquad (8a)$$

One checks rather easily from the comparison of (7*a*) and (8*a*) that the continuity condition (4) is satisfied provided that

$$\sin u_{\rm t} = \sin u_{\rm i} \qquad k_{\rm t}m = k_{\rm i} \qquad (mx' - x)_{z=0} = 0. \qquad (9)$$

So, when a TM FWM crosses the boundary $z = 0$, its direction $u_{\rm i}$ is left unchanged but it undergoes a frequency jump $\omega_1 = k_{\rm i}c \Rightarrow \omega_\tau = k_{\rm t}c = m^{-1}\omega_1$ and a lateral shift $x \Rightarrow x'$.

Substituting n for m into (6) and (9) gives, respectively, the phase of the TE FWM and the refraction condition for a TE FWM,

$$\sin u_{\rm t} = \sin u_{\rm i} \qquad k_{\rm t}n = k_{\rm i} \qquad (nx' = x)_{x=0} = 0. \qquad (9a)$$

At the output $z = d$ of the crystal slab, continuity requirements are satisfied with relations dual to (9) and (9*a*) (so that the phase Ω is the same for the transmitted and incident FWM which recovers its identity). However, taking into account dispersion, the matching condition for frequency becomes $m(\omega)\omega = \omega_{\rm i}$ and this equation may have zero (total reflection), one or several real solutions or even complex solutions supplying evanescent TM FWMs. One has a similar result for TE FWMs with $n(\omega)$.

As an illustration, let us take $\varepsilon = 1 - \omega_{\rm c}^2(\omega^2 - \omega\omega_{\rm b})^{-1}$ in which $\omega_{\rm b,c}$ are constant (such an expression can be used in the ionosphere [13]). Assuming $\mu = 1$, the frequency condition becomes

$$[1 - \omega_{\rm c}^2(\omega^2 + \omega\omega_{\rm b})^{-1}]^{1/2}\omega = \omega_{\rm i} \qquad (10)$$

supplying the cubic equation

$$\omega^3 + \omega_{\rm b}\omega^2 - (\omega_{\rm c}^2 + \omega_0^2)\omega - \omega_{\rm i}^2\omega_{\rm b} = 0 \qquad (10a)$$

with zero, one or three real solutions.

So, the propagation within the crystal is not the same for TM and TE FWMs, in particular, the velocity of the wavefront $\boldsymbol{v} = \operatorname{grad} \Omega/(\partial_t \Omega)$ is different since one has according to (3*a*) and (3*b*),

$$m^{-2}v_x^2 + n^{-2}v_z^2 = c^2 \qquad v_x^2 + v_z^2 = n^2c^2 \qquad (11)$$

for TM and TE FWMs, respectively.

Remark. The conditions for refraction of FWMs are different from that for harmonic plane waves due to the nonlinearity of their phase. However, when the parameter a tends to infinity, we find from (5) and (6) the phases $\Omega_{\rm i} = k_{\rm i}(ct - Z_{\rm i})$, $\Omega_{\rm t} = k_{\rm t}(ct - Z_{\rm t})$, of harmonic plane waves so that the continuity condition (4) becomes $k_{\rm i} \sin u_{\rm i} = k_{\rm t} \sin u_{\rm t}$ supplying the Descartes–Snell law if one disregards *a priori*, any possibility of a frequency jump. Mathematically, one could have as a solution $m \sin u_{\rm t} = p \sin u_{\rm i}$, $pk_{\rm t} = k_{\rm i}$, where p is an arbitrary real number such as $m^{-1}p \sin u_{\rm i} \leqslant 1$.

3. Refraction of conventional focus wave modes

To describe an electromagnetic FWM incident from an arbitrary direction on the $z = 0$ face of the crystal slab, we need the azimuthal angle v_i in addition to the polar angle u_i. Then, the phase Ω_i in vacuum is still given by (5) with (5*a*) replaced by

$$Z_i = r\sin u_i + z\cos u_i \qquad X_i = r\cos u_i + z\sin u_i \qquad r = x\cos v_i + y\sin v_i. \tag{12}$$

The phase Ω_t of the extraordinary FWM in the dielectric slab is a solution of equation (3*a*), the expression (6) is still valid with

$$Z_t = mr'\sin u_t + nz\cos u_t \qquad X_t = mr'\cos u_t - nz\sin u_t \qquad r' = x'\cos v_t + y'\sin v_t \tag{13}$$

in which, for the same reasons as previously, we have introduced the system of coordinates (x', y', z). Changing m into n in (13) gives the functions X, Z corresponding to the phase of the ordinary FWM solution of equation (3*b*).

Substituting (12) into (5), we find for the phase of the incident field

$$\Omega_i = k_i[a(ct - r\sin u_i - z\cos u_i) + c^2t^2 - r^2 - z^2](a + ct + r\sin u_i + z\cos u_i)^{-1} \tag{14}$$

and a simple calculation gives

$$(\partial_x\Omega_i)_{z=0} = -k_i\cos v_i(2r + a\sin u_i)(a + ct + r\sin u_i)^{-1} - k_i\sin u_i\cos v_i[a(ct - r\sin u_i) + c^2t^2 - r^2](a + ct + r\sin u_i)^{-2} \tag{14a}$$

$$(\partial_y\Omega_i)_{z=0} = -k_i\sin v_i(2r + a\sin u_i)(a + ct + r\sin u_i)^{-1} - k_i\sin u_i\sin v_i[a(ct - r\sin u_i) + c^2t^2 - r^2](a + ct + r\sin u_i)^{-2}. \tag{14b}$$

Substituting (13) into (6), we obtain

$$\Omega_t = k_t[a(ct - mr'\sin u_t - nz\cos u_t) + c^2t^2 - m^2r'^2 - n^2z^2] \times (a + ct + mr'\sin u_t + nz\cos u_t)^{-1} \tag{15}$$

$$(\partial_x\Omega_t)_{z=0} = -k_t m\cos v_t(2mr' + a\sin u_t)(a + ct + mr'\sin u_t)^{-1} - k_t m\sin u_t\cos v_t \times[a(ct - mr'\sin u_t) + c^2t^2 - m^2r'^2](a + ct + mr'\sin u_t)^{-2} \tag{15a}$$

$$(\partial_y\Omega_t)_{z=0} = -k_t m\sin v_t(2mr' + a\sin u_t)(a + ct + mr'\sin u_t)^{-1} - k_t m\sin u_t\sin v_t \times[a(ct - mr'\sin u_t) + c^2t^2 - m^2r'^2](a + ct + mr'\sin u_t)^{-2}. \tag{15b}$$

Substituting (14*a*), (15*a*) and (14*b*), (15*b*) into relations (4), one checks easily that the conditions for refraction into an extraordinary wave are a simple generalization of (9)

$$k_t m = k_i \qquad \sin u_t = \sin u_i \qquad \sin v_t = \sin v_i \qquad (mx' - x)_{z=0} = 0 \qquad (my' - y)_{z=0} = 0. \tag{16}$$

So, the direction of the incident FWM is left unchanged, while its frequency undergoes a jump and its position in the slab undergoes a lateral shift in both the x and y directions.

Substituting n for m into (16) gives the conditions for the refraction into an ordinary FWM,

$$k_t n = k_i \qquad \sin u_t = \sin u_i \qquad \sin v_t = \sin v_i \qquad (nx' - x)_{z=0} = 0 \qquad (nx' - x)_{z=0} = 0. \tag{16a}$$

All the discussion in the previous section holds valid, in particular the incident FWM, in spite of being divided within the slab into ordinary and extraordinary components, recovers its identity at the output of the crystal. The main difference, but an important one, is that ordinary and extraordinary FWMs propagate simultaneously, and we will have to look after their amplitude to know the part of the incident FWM that each of them carries away in the crystal. This work is made easier if we first consider the excitation of TM and TE FWMs.

4. Excitation of TM and TE focus wave modes

4.1. TM focus wave modes

Assuming the axes of coordinates in the directions of the principal axes of the permittivity tensor, Maxwell's equations for TM (H_y, E_x, E_z) electromagnetic waves are in the anisotropic crystal (for simplicity, we suppose $\mu = 1$)

$$\begin{aligned}&\partial_z H_y = -\varepsilon c^{-1}\partial_t E_x \qquad \partial_x H_y = \eta c^{-1}\partial_t E_z\\ &c^{-1}\partial_t H_y = \partial_x E_z - \partial_z E_x.\end{aligned} \tag{17}$$

The phase Ω of a TM FWM solution of equation (17) [11] is (with $m = \eta^{1/2}$, $n = \varepsilon^{1/2}$)

$$\Omega = k(ct - Z - gX^2) \qquad g = (a + ct + Z)^{-1} \tag{18}$$

$$Z = nz\cos u + mx'\sin u \qquad X = mx'\cos u - nz\sin u. \tag{18a}$$

Relations (17), (18) and (18*a*) are valid in free space with $\varepsilon = \eta = 1, m = n = 1$ and x' replaced by x. Then, using the subscripts i, r, t to label the quantities connected with incident, reflected and refracted waves and taking into account the conditions (9) for refraction and the Descartes–Snell laws for reflection, we obtain the relations

$$u_\mathrm{t} = u_\mathrm{i} \qquad u_\mathrm{r} = \pi - u_\mathrm{i} \tag{19a}$$

$$k_\mathrm{r} = k_\mathrm{i} = k_\mathrm{t} m \qquad (mx' - x)_{z=0} = 0. \tag{19b}$$

The FWM solution of equations (17) is obtained [11] by assuming that the component H_y of the magnetic field is a scalar FWM with the phase Ω, that is ($\mathrm{i} = \sqrt{-1}$)

$$H_y = B(g)^{1/2}\exp(\mathrm{i}\Omega) \tag{20}$$

in which g is expression (18) and B is an amplitude to be determined. Substituting (21) into the first two equations of the system (17) gives the components of the electric field in the form

$$E_{x,z} = Bg_{x,z}\exp(\mathrm{i}\Omega) \tag{21}$$

and to obtain $g_{x,z}$, we assume the wavenumber k to be large enough to make the derivatives of $g_{x,z}$ negligible with respect to the derivatives of Ω (high-frequency approximation). Then, substituting (21) and (22) into the first equation of the system (17) gives

$$(g)^{1/2}\partial_z\Omega = -\varepsilon g_x c^{-1}\partial_t\Omega \tag{22}$$

and using (18) a simple calculation gives

$$\begin{aligned}&c^{-1}\partial_t\Omega = k(1 + g^2X^2)\\ &\partial_z\Omega = kn[-\cos u(1 - g^2X^2) + 2\sin u\, gX].\end{aligned} \tag{23}$$

So, according to (22) and (23)

$$g_x = n(\varepsilon)^{-1}(g)^{1/2}[\cos u(1 - g^2X^2) - 2\sin u\, gX](1 + g^2X^2)^{-1} \tag{24}$$

which determines E_x in the frame of the high-frequency approximation. Changing $(\cos u, \sin u)$ into $(\sin u, -\cos u)$ and ε, n into η, m in (24) gives the attenuation factor g_z of E_z

$$g_z = m(\eta)^{-1}(g)^{1/2}[\sin u(1 - g^2X^2) + 2\cos u\, gX](1 + g^2X^2)^{-1}. \tag{24a}$$

Expressions (20)–(24*a*), obtained for a TM focus wave mode in the crystal, are also valid in free space with $\varepsilon = \eta = 1$ and $m = n = 1$. So, we know the expressions for the reflected and refracted focus wave modes that depend on the unknown amplitudes B_r and B_t of the component H_y to be obtained by using some boundary conditions on the surface of the slab.

These boundary conditions, demanding that across the interface $z = 0$ the tangential components H_y, E_x should be continuous, are

$$(H_{y,i} + H_{y,r} - H_{y,t})_{z=0} = 0 \tag{25a}$$

$$(E_{x,i} + E_{x,r} - E_{x,t})_{z=0} = 0. \tag{25b}$$

We need the expressions for H_y, E_x on $z = 0$. From (18*a*), taking into account (19*a*) and (19*b*) we obtain

$$(Z_i = Z_r = Z_t)_{z=0} = x \sin u_i \qquad (X_i = -X_r = X_t)_{z=0} = x \cos u_i \tag{26}$$

and using (18)

$$(\Omega_i = \Omega_r = \Omega_t m)_{z=0} = \boldsymbol{\Omega} \qquad (g_i = g_r = g_t)_{z=0} = \boldsymbol{g} \tag{27}$$

$$\boldsymbol{\Omega} = k_i(ct - x \sin u_i - \boldsymbol{g}x^2 \cos u_i) \qquad \boldsymbol{g} = (a + ct + x \sin u_i)^{-1}. \tag{27a}$$

Substituting (27) into (20) gives

$$\begin{aligned} &(H_{y,i})_{z=0} = B_i \boldsymbol{g}^{1/2} \exp(i\boldsymbol{\Omega}) \qquad (H_{y,r})_{z=0} = B_r \boldsymbol{g}^{1/2} \exp(i\boldsymbol{\Omega}) \\ &(H_{y,t})_{z=0} = B_t \boldsymbol{g}^{1/2} \exp(im^{-1}\boldsymbol{\Omega}). \end{aligned} \tag{28}$$

On the other hand, using (18), (19), (26) and (24), a simple calculation gives

$$(g_{x,i} = -g_{x,r} = \varepsilon(n)^{-1} g_{x,t})_{z=0} = \boldsymbol{g}_x \tag{29}$$

$$\boldsymbol{g}_x = \boldsymbol{g}^{1/2} \cos u_i (1 - \boldsymbol{g}^2 x^2 \cos^2 u_i - 2\boldsymbol{g}x \sin u_i)(1 + \boldsymbol{g}^2 x^2 \cos^2 u_i)^{-1} \tag{29a}$$

so that according to (21),

$$\begin{aligned} &(E_{x,i})_{z=0} = B_i \boldsymbol{g}_x \exp(i\boldsymbol{\Omega}) \qquad (E_{x,r})_{z=0} = -B_r \boldsymbol{g}_x \exp(i\boldsymbol{\Omega}) \\ &(E_{x,t})_{z=0} = n\varepsilon^{-1} B_t \boldsymbol{g}_x \exp(im^{-1}\boldsymbol{\Omega}). \end{aligned} \tag{30}$$

Substituting (28) and (30) into (25*a*) and (25*b*) gives

$$\begin{aligned} &B_i + B_r = B_t \exp[i\boldsymbol{\Omega}(m^{-1} - 1)] \\ &B_i - B_r = B_t n\varepsilon^{-1} \exp[i\boldsymbol{\Omega}(m^{-1} - 1)] \end{aligned} \tag{31}$$

and since $n = \sqrt{\varepsilon}$, $m = \sqrt{\eta}$ we get finally with $f(\eta) = \exp[i\boldsymbol{\Omega}(1 - \eta^{-1/2})]$

$$B_r = (\sqrt{\varepsilon} - 1)(\sqrt{\varepsilon} + 1)^{-1} B_0 \qquad B_t = 2\sqrt{\varepsilon}(\sqrt{\varepsilon} - 1)^{-1} B_0 f(\eta) \tag{31a}$$

where B_r and B_t are the Fresnel coefficients for the reflection and transmission of a TM FWM incident on a dielectric uniaxial crystal.

4.2. TE focus wave modes

Maxwell's equations for TE (E_y, H_x, H_z) electromagnetic waves are in the anisotropic crystal, still supposing $\mu = 1$

$$\partial_z E_y = c^{-1}\partial_t H_x \qquad \partial_x E_y = -c^{-1}\partial_t H_x$$
$$\varepsilon c^{-1}\partial_t E_y = \partial_z H_x - \partial_x H_z. \tag{32}$$

The dielectric constant η does not intervene in this equation. It was shown in [11] that the phase and the attenuation factor of the TE FWM, that we write Ω_n and g_n, are still given by (18) with $m = n$ in (18*a*). In addition, changing H_y, E_x, E_z, into εE_y, $-H_x$, $-H_z$, and making $\eta = 1$ transform the Maxwell equations (17) into the Maxwell equations (32). So, introducing an arbitrary amplitude C, the solution of equations (32) has the form

$$E_y = \varepsilon C(g_n)^{1/2}\exp(\mathrm{i}\Omega_n)$$
$$H_{x,z} = C g_{n,x,z}\exp(\mathrm{i}\Omega_n) \tag{33}$$

with $g_{n,x}$ and $g_{n,z}$ deduced from (24) and (24*a*) by changing g into g_n and making $\eta = 1$.

The solution (33) obtained for a TE FWM in the crystal is also valid with $n = 1$ and $\varepsilon = 1$ in free space, in particular for the incident and reflected TE FWMs. So, we have three amplitudes C_i, C_r, C_t, and to obtain the two unknowns C_r and C_t, we use the boundary conditions

$$(E_{y,\mathrm{i}} + E_{y,\mathrm{r}} - E_{y,\mathrm{t}})_{z=0} = 0 \qquad (H_{x,\mathrm{i}} + H_{x,\mathrm{r}} - H_{x,\mathrm{t}})_{z=0} = 0. \tag{34}$$

So we need the expressions of E_y and H_x on $z = 0$. The relations (26), (27) and (27*a*) are still valid with m replaced by n. Then,

$$(E_{y,\mathrm{i}})_{z=0} = C_\mathrm{i}\boldsymbol{g}^{1/2}\exp(\mathrm{i}\boldsymbol{\Omega}) \qquad (E_{y,\mathrm{r}}) = C_\mathrm{r}\boldsymbol{g}^{1/2}\exp(\mathrm{i}\boldsymbol{\Omega})$$
$$(E_{y,\mathrm{t}})_{z=0} = \varepsilon^{-1}C_\mathrm{t}\boldsymbol{g}^{1/2}\exp(\mathrm{i}n^{-1}\boldsymbol{\Omega}). \tag{35}$$

Similarly, relations (29) and (29*a*) hold valid with $m = n$ for g_n, so

$$(H_{x,\mathrm{i}})_{z=0} = C_\mathrm{i}\boldsymbol{g}_x\exp(\mathrm{i}\boldsymbol{\Omega}) \qquad (H_{x,\mathrm{r}})_{z=0} = C_\mathrm{r}\boldsymbol{g}_x\exp(\mathrm{i}\boldsymbol{\Omega})$$
$$(H_{x,\mathrm{t}})_{z=0} = n\varepsilon^{-1}c\boldsymbol{g}_x\exp(\mathrm{i}n^{-1}\boldsymbol{\Omega}). \tag{36}$$

Substituting (35) and (36) into (34) gives the relations

$$C_\mathrm{i} + C_\mathrm{r} = C_\mathrm{t}\varepsilon^{-1}\exp[\mathrm{i}\Omega(n^{-1} - 1)]$$
$$C_\mathrm{i} - C_\mathrm{r} = n\varepsilon^{-1}C_\mathrm{t}\exp[\mathrm{i}\Omega(n^{-1} - 1)] \tag{37}$$

with the solution in which f is the function defined previously,

$$C_\mathrm{r} = (1 - \sqrt{\varepsilon})(1 + \sqrt{\varepsilon})^{-1}C_\mathrm{i} \qquad C_\mathrm{t} = 2\varepsilon(1 + \sqrt{\varepsilon})^{-1}f(\varepsilon)C_\mathrm{i}. \tag{38}$$

The Fresnel coefficients of TM and TE FWMs do not depend on the angle of incidence. Except for the factor f they have the same expressions as the Fresnel coefficients for the reflection of a plane wave at normal incidence [14].

5. Excitation of ordinary and extraordinary focus wave modes

5.1. General formulation

The conventional FWMs [1, 2] depend on $(x^2 + y^2)$ with g as an attenuation factor. However, there also exists [15] a second kind of FWMs that depend on $(x\cos v + y\sin v)^2$ with $g^{1/2}$ as an attenuation factor. We work here with these last FWMs, since they reduce to TM and TE FWMs for $v = 0$, making calculations easier. Note that an incident FWM generates four waves: the ordinary and extraordinary waves within the crystal and the corresponding reflected fields. We still use the subscripts i, r, t, to denote the incident, reflected and refracted waves, but now r and t are the sets (r_o, r_e) and (t_o, t_e). For simplicity we use r and t, when no confusion is possible.

The phase Ω of an extraordinary FWM in the crystal is [15]

$$\Omega = k(ct - Z - gX^2) \qquad g = (a + ct + Z)^{-1} \tag{39}$$

$$Z = mr\sin u + nz\cos u \qquad X = mr\cos u - nz\sin u \qquad r = x'\cos v + y'\sin v \tag{39a}$$

with $n = \sqrt{\varepsilon}$, $m = \sqrt{\eta}$ and using the coordinates x', y', as previously.

Changing m into n in (39a) gives the phase of the ordinary wave, while in free space for the incident and reflected waves $m = n = 1$.

According to (16) and (16a) the conditions for reflection and refraction into an extraordinary wave are

$$u_t = u_i \qquad u_r = \pi - u_i \qquad v_i = v_r = v_t \tag{40a}$$

$$k_i = k_r = mk_t \qquad (mx' - x)_{z=0} = 0 \qquad (my' - y)_{z=0} = 0 \qquad r = r_e \qquad t = t_e. \tag{40b}$$

For the ordinary wave the conditions (40b) become

$$k_i = k_r = nk_t \qquad (nx' - x)_{z=0} = 0 \qquad (ny' - y)_{z=0} = 0 \qquad r = r_o \qquad t = t_o. \tag{40c}$$

We need the expressions of Ω and g on the boundary surface $z = 0$. From (39a) and from the conditions (40a)–(40c), we get with $r_i = x\cos v_i + y\sin v_i$,

$$(Z_{i,r,t})_{z=0} = r_i\sin u_i \qquad (X_{i,t})_{z=0} = r_i\cos u_i \qquad (X_r)_{z=0} = -r_i\cos u_i \tag{41}$$

and substituting (41) into (39)

$$(\Omega_i = \Omega_r = m\Omega_{te} = n\Omega_{to})_{z=0} = \chi \qquad (g_{i,r,t})_{z=0} = \gamma \tag{42}$$

$$\chi = k(ct - r_i\sin u_i - \gamma r_i^2\cos 2u_i) \qquad \gamma = (a + ct + r_i\sin u_i)^{-1}. \tag{42a}$$

Now, we look for the extraordinary wave solutions of Maxwell's equations within the crystal in the following form, the subscript j takes the values 1, 2, 3, corresponding to the coordinates x, y, z,

$$E_j = Ea_j\exp(\mathrm{i}\Omega_s) \qquad H_j = Eb_j\exp(\mathrm{i}\Omega) \tag{43}$$

in which E is an amplitude to be determined and Ω is the phase (39). We assume the derivatives of a_j, b_j to be negligible with respect to the derivatives of Ω (high frequency approximation). So, with $\partial_\beta = (\partial_j, \partial_0)$, $\partial_j = \partial/\partial x_j$, $\partial_0 = c^{-1}\partial/\partial t$

$$\partial_\beta E_j = \mathrm{i}\partial_\beta\Omega E_j \qquad \partial_\beta H_j = \mathrm{i}\partial_\beta\Omega H_j \qquad j = 1, 2, 3 \tag{44}$$

and we introduce the notation

$$w_j = \partial_j \Omega \qquad w_0 = c^{-1}\partial_t \Omega. \tag{44a}$$

Then, with (44) and (44*a*) the Maxwell equations become

$$\begin{aligned} &w_y b_z - w_z b_y = \varepsilon w_0 a_x \qquad w_z b_x - w_x b_z = \varepsilon w_0 a_y \qquad w_x b_y - w_y b_x = \eta w_0 a_z \\ &w_y a_z - w_z a_y = -w_0 b_x \qquad w_z a_x - w_x a_z = -w_0 b_y \qquad w_x a_y - w_y a_x = -w_0 b_z. \end{aligned} \tag{45}$$

Since the phase Ω is a solution of the characteristic equation of Maxwell's equations, the homogeneous system (45) has an infinity of solutions which can be expressed in terms of one of its unknowns, say a_3. Then, to obtain a FWM solution of Maxwell's equations with the phase Ω, we choose a_3 so that E_z is a scalar FWM, that is $a_3 = g^{1/2}$ with g given by (39).

These results obtained for the extraordinary wave are also valid for the ordinary wave with $m = n$ and for the incident and reflected waves for $m = n = 1$. Similarly to (43), we write the ordinary wave solution of Maxwell's equations

$$E_j = E_n a_{n,j} \exp(\mathrm{i}\Omega_n) \qquad H_{nj} = E_n b_{n,j} \exp(\mathrm{i}\Omega_n). \tag{46}$$

So, an incident field with amplitude E_{i} generates four waves with the unknown amplitudes E_{r}, E_{t}, $(r = r_{\mathrm{e}}, r_{\mathrm{o}})$, $(t = t_{\mathrm{e}}, t_{\mathrm{o}})$ which can be obtained by imposing some boundary conditions on the interface $z = 0$. These conditions require the continuity of the tangential components of $\boldsymbol{E}$ and $\boldsymbol{H}$, that is

$$[(\boldsymbol{H}_{\mathrm{i}} + \boldsymbol{H}_{\mathrm{r}} - \boldsymbol{H}_{\mathrm{t}})_{x,y}]_{z=0} = 0 \qquad [(\boldsymbol{E}_{\mathrm{i}} + \boldsymbol{E}_{\mathrm{r}} - \boldsymbol{E}_{\mathrm{t}})_{x,y}]_{z=0} = 0. \tag{47}$$

However, solving the system (47) of four equations for the four unknown amplitudes is a formidable task, so we shall proceed differently by using the results of section 4.

5.2. TE, TM formulation

According to (31) and (38), the coefficients of reflection B_{r}, C_{r}, do not depend on the angle u which suggests taking $u = 0$, while the conditions for reflection and refraction impose $v_{\mathrm{t}} = v_{\mathrm{r}} = v_{\mathrm{i}} = v$. From now on, a double primed quantity denotes an expression in which $u = 0$. Then, using (39*a*) the phase and the attenuation factor of the extraordinary focus wave mode become with $X = x\cos v + y \sin v$

$$\Omega'' = k(ct - nz - g'' m^2 X^2) \qquad g'' = (a + ct + nz)^{-1}. \tag{48a}$$

Setting $m = n$ gives the phase and the attenuation factor Ω''_n, g''_n, of the ordinary FWM

$$\Omega''_n = k(ct - nz - g'' n^2 X^2) \qquad g''_n = g''. \tag{48b}$$

Then, using the coordinates X, $Y = x \sin v - y \cos v$, and the following components of the electromagnetic field:

$$E_X = E_x \cos v + E_y \sin v \qquad H_Y = H_x \sin v - H_y \cos v \qquad E_z \tag{49a}$$

$$H_X = H_x \cos v + H_y \sin v \qquad E_X = E_x \sin v - E_y \cos v \qquad H_z. \tag{49b}$$

One checks at once that (E_X, E_z, H_Y) with the phase Ω'' is a TM field (note that Ω'' and Ω''_n do not depend on Y) satisfying the Maxwell equations (17) with ∂_x changed into ∂_X. Similarly, (H_X, H_z, E_Y) with the phase Ω''_n is a TE field solution of equations (32).

Then, using (43), we may write (49*a*)

$$H_Y = b_Y E\exp(\mathrm{i}\Omega'') \qquad E_X = a_X E\exp(\mathrm{i}\Omega'') \qquad E_z = a_z E\exp(\mathrm{i}\Omega'') \tag{50}$$

$$b_Y = b_x\sin v - b_y\cos v \qquad a_X = a_x\cos v + a_y\sin v \tag{50a}$$

while from (46) and (49*b*) we obtain

$$E_Y = a_Y E_n\exp(\mathrm{i}\Omega''_n) \qquad H_X = b_X E_n\exp(\mathrm{i}\Omega''_n) \qquad H_z = b_z E_n\exp(\mathrm{i}\Omega''_n) \tag{51}$$

$$a_Y = a_x\sin v - a_y\cos v \qquad b_X = b_x\cos v + b_y\sin v. \tag{51a}$$

These fields depend on two unknown amplitudes E and E_n. From now on, we concentrate on the components E_z and H_z. Identifying (50) and (51) with the TM and TE FWMs of section 4, we have according to (27) and (33),

$$\begin{aligned} E_z &= B_{\mathrm{t}}[g_z\exp(\mathrm{i}\Omega)]_{u=0} \\ H_z &= C_{\mathrm{t}}[g_{n,z}\exp(\mathrm{i}\Omega_n)]_{u=0} \end{aligned} \tag{52}$$

in which Ω and g_z are given by (18) and (24*a*) which also supply Ω_n and $g_{n,z}$ by changing m into n, while B_{t} and C_{t} are the Fresnel coefficients (31*a*) and (38). So from (50)–(52) we obtain the relations that determine E and E_n in terms of B_{t} and C_{t}, that is finally in terms of the incident amplitudes B_{i} and C_{i},

$$\begin{aligned} a_z E\exp(\mathrm{i}\Omega'') &= B_{\mathrm{t}}[g_z\exp(\mathrm{i}\Omega)]_{u=0} \\ b_z E_n\exp(\mathrm{i}\Omega'') &= C_{\mathrm{t}}[g_{n,z}\exp(\mathrm{i}\Omega)]_u = 0. \end{aligned} \tag{53}$$

However, in section 4, the TM and TE fields are two independent sets so that the amplitudes B_{i} and C_{i} are arbitrary. The situation is different here since the fields (50) and (51) are connected through relations (49) and one has to determine B_{i} and C_{i} in terms of the amplitude E_{i} of the incident electromagnetic field. We remind the reader that quantities connected with the incident field are deduced from quantities connected with the extraordinary field (respectively, ordinary field) by making $m = n = 1$ (respectively $n = 1$). So, we find from (50) and (51)

$$\begin{aligned} E_{\mathrm{i},z} &= E_{\mathrm{i}}W_z \qquad & W_z &= [a_z\exp(\mathrm{i}\Omega'')]_{m=n=1} \\ H_{\mathrm{i},z} &= E_{\mathrm{i}}W_{n,z} \qquad & W_{n,z} &= [b_z\exp(\mathrm{i}\Omega''_n)]_{n=1} \end{aligned} \tag{54}$$

while relations (52) give

$$\begin{aligned} E_{\mathrm{i},Z} &= B_{\mathrm{i}}W_Z \qquad & W_z &= [g_Z\exp(\mathrm{i}\Omega)]_{u=0,m=n=1} \\ H_{\mathrm{i},Z} &= C_{\mathrm{i}}W_{n,Z} \qquad & W_{n,z} &= [g_{n,Z}\exp(\mathrm{i}\Omega_n)]_{u=0,n=1} \end{aligned} \tag{55}$$

so that from (54) and (55)

$$B_{\mathrm{i}} = E_{\mathrm{i}}W_z/W_Z \qquad C_{\mathrm{i}} = E_{\mathrm{i}}W_{n,z}/W_{n,Z}. \tag{56}$$

So, we have obtained the amplitudes of the ordinary and extraordinary FWMs excited in a uniaxial dielectric crystal by a FWM incident normally on the face $z = 0$ of the crystal. However, as stated previously, this result does not depend on the angle of incidence. Consequently, one may generalize the expressions obtained in this section to any angle $u \in [0, \pi/2)$ only by changing double primed into unprimed quantities.

6. Discussion

We must first insist that in this series of papers on FWMs and crystals, we used a particular kind of FWMs with the particular property of being easy to split into TE and TM components. This is not the case for conventional FWMs [1, 2], for which propagation in anisotropic media as well as their excitation is still an open problem.

FWMs (whether conventional or not) have a nonlinear phase which makes the analysis of their behaviour and of their properties in many situations difficult when, for instance, scattering and diffraction intervene. However, a counterpart to this nonlinearity is the simplification of the conditions for refraction. Then, a dielectric uniaxial crystal appears as a separator of the incident FWM into its TM and TE components, each propagating unperturbed in the crystal except for a frequency jump and a lateral shift at the input. At the output of the crystal, the two components rejuvenate the incident FWM with a decreased amplitude so that a uniaxial crystal is an attenuator of FWMs.

This behaviour contrasts with that of plane waves where the relative (with respect to FWM) complexity of Descartes–Snell laws and Fresnel coefficients that depend of the angle of incidence, generates a great wealth of phenomena which have given rise to important works [3, 14, 16–18].

One may wonder what would have happened if Nature had required its description to be in terms of FWMs rather than plane waves.

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